International Journal of Theoretical Physics, Vol. 9, No. 1 (1974), pp. 59–68

Singularities in Weyl Gravitational Fields†

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Received: 27 February 1973

Abstract

The peculiarities of the scalar $S \equiv R_{ijkl}R^{ijkl}$ are exhibited for two axially-symmetric static (Weyl) gravitational fields. By examining S along curved families of trajectories to the Weyl singularities, examples are found which contradict previous claims by Gautreau and Anderson regarding 'directional singularities'. Proper circumferences about the Bach and Weyl line-mass singularity are also examined. There is no apparent correlation between the source structure and the behaviour of S from this analysis.

1. *Introduction*

The concept of a singularity in general relativity has been a subject of much interest and debate over the years. The astrophysical interest in gravitational collapse and the nature of the evolving cosmological models has lent new weight and purpose to the study of singularities. The interest and concern has led a number of investigators to seek clear-cut criteria for intrinsic singularities and to examine their properties. It may emerge that no single rule for singularities is possible and that one must look for one of several indicators.

Perhaps the most universal approach to locating intrinsic singularities in gravitational fields has been the search for infinities in the invariant scalars formed from combinations of covariant elements of the theory (Gautreau & Anderson, 1967). This approach has been criticised by Geroch (1968) on the grounds that the invariant tetrad components of the Riemann tensor can be altered and in fact made infinite at will by the proper choice of tetrad. This criticism can be countered by questioning the status of tetrad vectors as basic elements of the theory. Moreover, one could argue that one should confine one's attention to scalars such as the Kretschmann scalar,

$$S \equiv R_{ijkl}R^{ijkl} \tag{1.1}$$

where R_{ijkl} is the Riemann tensor, which are invariant in every sense.

† Supported in part by the National Research Council of Canada, Grant No. A5340.
‡ National Research Council of Canada Predoctoral Fellow.

Since the Riemann tensor fundamentally characterises the gravitational field, it seems reasonable to say that if S becomes infinite, or indeed indeterminate at some point or region in the space-time, then one has located an intrinsic singularity. However, the converse is not the case. An intrinsic singularity will be exhibited for which S is very well behaved.

A few years ago, Gautreau & Anderson (1967) developed the concept of a 'directional singularity' noting that 'the singular behaviour of an invariant scalar may not always unambiguously indicate the location of an intrinsic singularity of a gravitational field'. They showed that for the field of a Curzon (1924) particle, $S \to \infty$ for every straight line trajectory to the origin except along the z-axis where $S \to 0$ 'indicating that the origin is not the location of an intrinsic singularity. In other words, there appears to be a directionality associated with the "singularity" at the origin.'

Evidently, the meaning which they wished to convey by their analysis was misconstrued by Bonnor & Sackfield (1968) who cited their work as evidence that scalar invariants need not become infinite at the location of intrinsic singularities. In a rejoinder, Gautreau & Hoffman (1969) affirmed that it is the infinities of the scalar invariants which locate the intrinsic singularities of a gravitational field! According to the proponents of the directional singularity concept, one does not meet an intrinsic singularity by coming to the origin in the direction of the z-axis but one does meet one for any other direction of approach. The question of the actual location of the intrinsic singularity is left in an aura of doubt. Perhaps that was the original intention.

In this paper, we demonstrate that the concept of 'directionality' with regard to singularities has no foundation, thereby removing the intuitively disturbing proposal that one precise direction should have a completely distinct qualitative character from that of every other direction vis-à-vis the character of the singularity. We also present a clear-cut example of an intrinsic singularity for which the scalar S is well-behaved in every sense in a region which has always been, justifiably, regarded as intrinsically singular.

In Section 2, the Curzon particle singularity is re-examined. By evaluating the scalar S along power-law trajectories, an interesting set of possibilities is revealed. In particular, there is a range of trajectories which approach the Curzon particle via the z-axis direction and which yield an infinite limit for S rather than the zero limit by approaching along the z-axis (the $r = 0$ trajectory). By the criterion of Gautreau and Anderson, one might be led to call the termination point in the z-axis direction both singular and non-singular. We feel that it is more reasonable to simply call it singular along with every other termination point. The field of two Curzon particles is also considered.

In Section 3, the Weyl field generated by the Newtonian potential of a constant density line mass is examined. At the end point, the limiting value for S depends on the line density and trajectory. The proper circumferences are also evaluated.

Concluding remarks follow in Section 4.

2. *The Curzon Particle Singularity*

Weyl (1917, 1919; see also Synge, 1960) showed that static, axially-symmetric gravitational fields can be expressed by the metric

$$ds^2 = e^{2\lambda} dt^2 - e^{2(\nu-\lambda)}(dr^2 + dz^2) - r^2 e^{-2\lambda} d\phi^2 \tag{2.1}$$

where†

$$\lambda_{11} + \lambda_{22} + \lambda_1/r = 0 \tag{2.2}$$

$$\nu_1 = r(\lambda_1{}^2 - \lambda_2{}^2), \qquad \nu_2 = 2r\lambda_1\lambda_2 \tag{2.3}$$

are the equations satisfied by λ and ν in vacuum. Equation (2.2) is precisely Laplace's equation in cylindrical polar coordinates and hence axially symmetric Newtonian fields can be used to generate general relativistic gravitational fields. There is a lack of correspondence, however. For example, the Newtonian potential of a spherically symmetric mass distribution

$$\lambda = -\frac{m}{\rho} = -\frac{m}{\sqrt{(r^2+z^2)}} \tag{2.4}$$

generates the Curzon metric (1924)

$$ds^2 = e^{-2m/\rho} dt^2 - e^{(2m/\rho - m^2r^2/\rho^4)}(dr^2 + dz^2) - e^{2m/\rho} r^2 d\phi^2 \tag{2.5}$$

which is not equivalent to the Schwarzschild solution, the unique static, spherically symmetric vacuum solution.

Gautreau & Anderson (1967) computed the scalar S for the Curzon metric,

$$S = \exp\left[\frac{2m}{\rho}\left(\frac{mr^2}{\rho^3} - 2\right)\right]\left\{\text{polynomial in } \frac{1}{\rho}\right\} \tag{2.6}$$

Since $S \to \infty$ as $\rho \to 0$ for every straight line trajectory except for an approach along the z-axis ($r = 0$) where $S \to 0$, they concluded that the direction of the z-axis is somewhat special and hence the 'directional' concept.

The situation is considerably more complicated than this (Cooperstock *et al.*, 1972). The scalar S can just as readily be evaluated along curved trajectories. Consider the family of trajectories

$$z = Cr^n, \qquad n > 0 \tag{2.7}$$

where, for simplicity, C is taken to be positive and $z \to 0_+$ to the origin. The results are tabulated in Table 1.

From the standpoint of the trajectories $2/3 < n < 1$, the direction of the z-axis is not very special at all in that the scalar S becomes infinite as the singularity is approached. Clearly, the value of the scalar depends on the trajectory as well as the direction which is attained. Moreover, for the critical trajectory $n = 2/3$, the scalar depends on the relationship between the mass of the Curzon particle and the trajectory parameter C.

At this stage, it is reasonable to consider the true significance of the value

† 1, 2, 3, denote r, z, ϕ respectively and λ_1 denotes $\partial\lambda/\partial r$, etc. $G = c = 1$.

TABLE 1. Limits for S in the Curzon metric

Range	S	Direction as $r \to 0$
$0 < n < 2/3$	0	z-axis
$n = 2/3 \quad C < \left(\frac{m}{2}\right)^{1/3}$	∞	z-axis
$n = 2/3 \quad C \geqslant \left(\frac{m}{2}\right)^{1/3}$	0	z-axis
$2/3 < n < 1$	∞	z-axis
$n = 1 \quad r \neq 0$	∞	all directions other than the z-axis
$r = 0$ (the z-axis trajectory)	0	z-axis
$n > 1$	∞	r-axis

of S with regard to singularities. Clearly, one would want to call a point 'singular' if S were, in some way, infinite there. The suggestion of Gautreau and Anderson is that one should qualify the description if S should have 'directional' properties. Implicit in their work is the conviction that the infinite value of S is not merely a sufficient condition but also a necessary condition for the existence of an intrinsic singularity. If this were indeed the case, then the subtleties in the behaviour of S would be a subject worthy of considerable study. One would be motivated, by the foregoing results, to introduce the concept of a 'trajectory singularity' since the behaviour of S is trajectory dependent. However, the infinite value of S is not a necessary condition for an intrinsic singularity. This point has been stressed before (Rindler, 1969; Bonnor & Sackfield, 1968).

A very straightforward example of this phenomenon is the field of two separated Curzon particles. The metric was derived by Silberstein (1936) and corrected by Einstein & Rosen (1936). The corrected metric has a line singularity between the two particles which plays the role of a strut, holding the particles at a fixed separation. However, the scalar S is readily shown to be finite on this line and for every approach to this line which avoids the end points.

3. *Metric of Bach and Weyl*

Applying the Weyl formalism Bach & Weyl (1922; see also, Robertson & Noonan, 1968) derived the metric which is generated by the Newtonian potential of a constant density line mass,

$$\lambda = \frac{\mu}{2l} \ln \frac{R_1 + R_2 - 2l}{R_1 + R_2 + 2l} \tag{3.1}$$

where

$$R_1{}^2 = (z - l)^2 + r^2$$
$$R_2{}^2 = (z + l)^2 + r^2 \tag{3.2}$$

μ is the mass of the line and $2l$ is its length in the Newtonian picture. The Weyl equations (2.3) yield

$$\nu = \frac{1}{2}\left(\frac{\mu}{l}\right)^2 \ln \frac{(R_1 + R_2)^2 - 4l^2}{4R_1 R_2} \tag{3.3}$$

For the special case

$$\frac{\mu}{l} \equiv \psi = 1 \tag{3.4}$$

the coordinate transformation

$$\rho = \frac{R_1 + R_2 + 2\mu}{2}$$
$$\cos\theta = \frac{R_2 - R_1}{2\mu} \tag{3.5}$$

casts the metric into the form of the Schwarzschild solution

$$ds^2 = \left(1 - \frac{2\mu}{\rho}\right) dt^2 - \left(1 - \frac{2\mu}{\rho}\right)^{-1} d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\phi^2) \tag{3.6}$$

where the line $-l \leqslant z \leqslant l$ maps into the sphere $\rho = 2\mu$.

Using differential forms (Israel, 1970) in conjunction with coordinates ξ, η defined by

$$R_1 + R_2 = 2l \cosh\xi$$
$$R_2 - R_1 = 2l \cos\eta \tag{3.7}$$

with basis 1-forms

$$\theta^1 = \left(\tanh\frac{\xi}{2}\right)^{\psi} dt$$
$$\theta^2 = l\left(\coth\frac{\xi}{2}\right)^{\psi} (\sinh\xi)^{\psi^2} (\sinh^2\xi + \sin^2\eta)^{\frac{1-\psi^2}{2}}\, d\xi$$
$$\theta^3 = l\left(\coth\frac{\xi}{2}\right)^{\psi} (\sinh\xi)^{\psi^2} (\sinh^2\xi + \sin^2\eta)^{\frac{1-\psi^2}{2}}\, d\eta$$
$$\theta^4 = l\left(\coth\frac{\xi}{2}\right)^{\psi} \sinh\xi \sin\eta\, \mathrm{d}\phi \tag{3.8}$$

the Bach and Weyl line element becomes

$$ds^2 = (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2 \tag{3.9}$$

The frame components of the Riemann tensor $R^A{}_{BCD}$ are given in the Appendix and the scalar S is

$$\begin{aligned} S &= R^{ABCD} R_{ABCD} \\ &= 4[(R^1{}_{212})^2 + (R^1{}_{213})^2 + (R^1{}_{312})^2 \\ &\quad + (R^1{}_{313})^2 + (R^1{}_{414})^2 + (R^2{}_{323})^2 + (R^2{}_{424})^2 \\ &\quad + (R^3{}_{424})^2 + (R^3{}_{434})^2] \end{aligned} \tag{3.10}$$

S was evaluated for two straight line trajectories approaching the end point $z = l$, $r = 0$: the z-axis trajectory $r = 0$, $z \to l_+$ and the trajectory $z = l$, $r \to 0_+$. In both cases, $S \to \infty$ for $0 < \psi < 2$ and $S \to 0$ for $\psi > 2$ in agreement with Gautreau & Anderson (1967) and Gautreau (1969).

The trajectories

$$z - l = Cr^n \tag{3.11}$$

were also considered. The results are given in Table 2. The peculiarity here exhibited by S is the onset of its infinity at $\psi = 3$ for the power law trajectories which approach the z-axis direction, rather than at $\psi = 2$, as in the case of straight line z-axis approach. Gautreau (1969) refers to the non-existence of intrinsic singularities as the end points are approached for $\psi > 2$. However, for $0 < n < 1$, S becomes infinite as the end points are approached for $2 < \psi < 3$ and hence even the most daring would be inclined to call the singularity 'intrinsic' for this range. Indeed, by considering more general families of trajectories, one can demonstrate that the ψ range which yields an infinite value for S can be extended to include all $\psi > 0$ (except $\psi = 1$, the Schwarzschild case).

TABLE 2. Limits for S in the metric of Bach and Weyl

n	ψ	S
$0 < n < 1$	$0 < \psi < 2$	∞
	$2 < \psi < 3$	∞
	$3 < \psi$	0
$n > 1$	$0 < \psi < 2$	∞
	$2 < \psi$	0

The similarity between this metric and the Curzon metric is the possibility for S to be infinite or zero for trajectories which approach the z-axis direction. However the details are quite different.

Gautreau (1969; see also, Stachel, 1968) postulated a correlation between the onset of directional singularities and the structure of the source as interpreted from the areas of equipotenial surfaces converging on the coordinate location of the rod. A perhaps more intuitive approach towards the determination of structure is the measure of proper circumferences along the

TABLE 3. Proper circumferences of the Bach and Weyl singularity

Trajectory	n	ψ	L
$z - l = Cr^n \quad C > 0$	$0 < n < 1$	$n\psi < 2$	0
$r \to 0_+$	$0 < n < 1$	$n\psi > 2$	∞
End point approached	$n > 1$	$\psi < 2$	0
	$n > 1$	$\psi > 2$	∞
$l - z = Cr^n \quad C > 0$	$0 < n < 1$	$(2 - n)\psi < 2$	0
$r \to 0_+$	$0 < n < 1$	$(2 - n)\psi > 2$	∞
End point approached	$n > 1$	$\psi < 2$	0
	$n > 1$	$\psi > 2$	∞
Interior points	Limits are	$\psi < 1$	0
$-l < z < l, r = 0$	independent of	$\psi = 1$	finite
approached	trajectory		Schwarzschild circumference
		$\psi > 1$	∞

'rod'. However, as in the case of the scalar S, one has the luxury of measuring the limiting values of circumferences along families of trajectories. The possibilities are even richer than one would imagine.

The values of the proper circumference L for the different cases are given in the following table (Table 3).

It is to be noted that the $n > 1$ trajectories to the end point yield the same limiting circumferences both for $z \to l_+$ and $z \to l_-$. However, for $n < 1$, the transition from zero to infinite circumferences occurs at $(2-n)\psi = 2$ for $z \to l_-$ and at $n\psi = 2$ for $z \to l_+$.

Stachel (1968) considered proper circumferences in his analysis of the Curzon particle. In the case of the Curzon particle the onsets of non-zero equipotential area and non-zero circumference are coincident. In the case of the Bach and Weyl solution this coincidence no longer exists. The onset of non-zero equipotential area is fixed at $\psi = 2$ whereas for $n < 1$ the value of ψ for the onset of non-zero circumference is dependent on the trajectory. Indeed, the ψ-value can be pushed to $\psi = 1$ for $z \to l_-$ in the limit $n \to 0$.

In the spirit of Gautreau and Stachel, one might be inclined to postulate a correlation between non-zero circumference and trajectory properties of a singularity by analogy with the postulated correlation between non-zero equipotential areas and directional singularities. This is particularly natural because the circumferences can be evaluated along the same trajectories as the scalar S. However, such a correlation is ruled out. As previously stated, it is possible to extend the range of ψ values for non-zero circumference to $\psi = 1$ whereas the line singularity exhibits no trajectory properties for $\psi < 2$.

4. *Concluding Remarks*

In this paper, we have demonstrated that the scalar $S = R_{ijkl}R^{ijkl}$, which is commonly used as a discriminator for intrinsic singularities in general relativistic fields has extremely varied properties in certain critical regions of two Weyl static axially symmetric fields. This great variability was achieved merely by changing from limits taken to the singular regions along straight line trajectories as in the works of Gautreau and Anderson, to simple power-law families of trajectories. In this manner, it was possible to find a counter-example to the 'directional' property attributed to the singularity of the Curzon particle. It was further demonstrated for the Bach and Weyl solution that the range of ψ-values for which S is infinite could be extended by considering power-law families of trajectories instead of straight line trajectories. It was indicated that this range could be extended to cover all $\psi > 0$ ($\psi \neq 1$) by consideration of more general trajectories. Moreover, the proper circumferences for the singular region of the line mass solution of Bach and Weyl were evaluated along the power-law trajectories and the possibilities were shown to be equally varied and interesting (albeit peculiar).

On the basis of these results, a correlation between non-zero proper circumference and trajectory dependence of the scalar S, was ruled out.

Whether or not the trajectory dependence of these quantities could be used to acquire important information about the singularities is an open question.

Appendix

The non-vanishing frame components of the Reimann tensor are

$$R^1{}_{221} = \frac{\psi \operatorname{sech}^2 \frac{\xi}{2}}{2D}\left\{- B \operatorname{csch} \xi + C\left[\frac{(\psi - 1)}{2} \operatorname{csch}^2 \frac{\xi}{2} - 1\right]\right\}$$

$$R^1{}_{321} = R^1{}_{231} = \frac{\psi(\psi^2 - 1)\sin 2\eta}{2D \sinh \xi}$$

$$R^1{}_{313} = -\frac{\psi B \operatorname{sech}^2 \frac{\xi}{2}}{2D \sinh \xi}$$

$$R^1{}_{414} = \frac{\psi AC \operatorname{sech}^2 \frac{\xi}{2}}{2D \sinh \xi}$$

$$R^3{}_{424} = R^2{}_{434} = \frac{\operatorname{sech}^2 \frac{\xi}{2}}{4D}[A(\psi^2 - 1)\sin 2\eta + 2(AC + B)\cot \eta]$$

$$R^3{}_{434} = \frac{\left[AB \operatorname{sech}^4 \frac{\xi}{2} + 4(1 - \psi^2)\cos^2 \eta + 4C\right]}{4D}$$

$$R^2{}_{323} = \frac{(\psi^2 - 1)}{CD}[C \cos 2\eta - \tfrac{1}{2}\sin^2 2\eta]$$
$$+ \frac{\psi}{D}\{2\cosh \xi - C \operatorname{csch}^2 \xi + \psi \sin^2 \eta \operatorname{csch}^2 \xi \cosh \xi\}$$
$$- \frac{\psi^2 \sin^2 \eta + [2\cosh^2 \xi + \sinh^2 \xi - \cosh \xi]}{D}$$
$$- \frac{B \operatorname{sech}^2 \frac{\xi}{2}\left[\operatorname{csch}\xi - \coth\xi - \frac{\sinh 2\xi}{C}\right]}{2D}$$

$$R^2{}_{424} = \left\{A \operatorname{sech}^4 \frac{\xi}{2}\left[(1 - \psi)\, C \coth \frac{\xi}{2} - B\right]\right.$$
$$\left. + 4C[\operatorname{csch}^2 \xi(\cosh \xi - \psi) - 1] + 4(\psi^2 - 1)\cos^2 \eta\right\} 4D$$

where

$$A = \frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi - \coth\frac{\xi}{2}\cosh\xi$$

$$B = \left(-\frac{\psi}{2}\operatorname{csch}^2\frac{\xi}{2}\sinh\xi + \psi^2\coth\frac{\xi}{2}\cosh\xi\right)C + (1-\psi^2)\coth\frac{\xi}{2}\sinh^2\xi\cosh\xi$$

$$C = \sinh^2\xi + \sin^2\eta$$

$$D = l^2\left(\coth\frac{\xi}{2}\right)^{2\psi}(\sinh\xi)^{2\psi^2}C^{2-\psi^2}$$

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